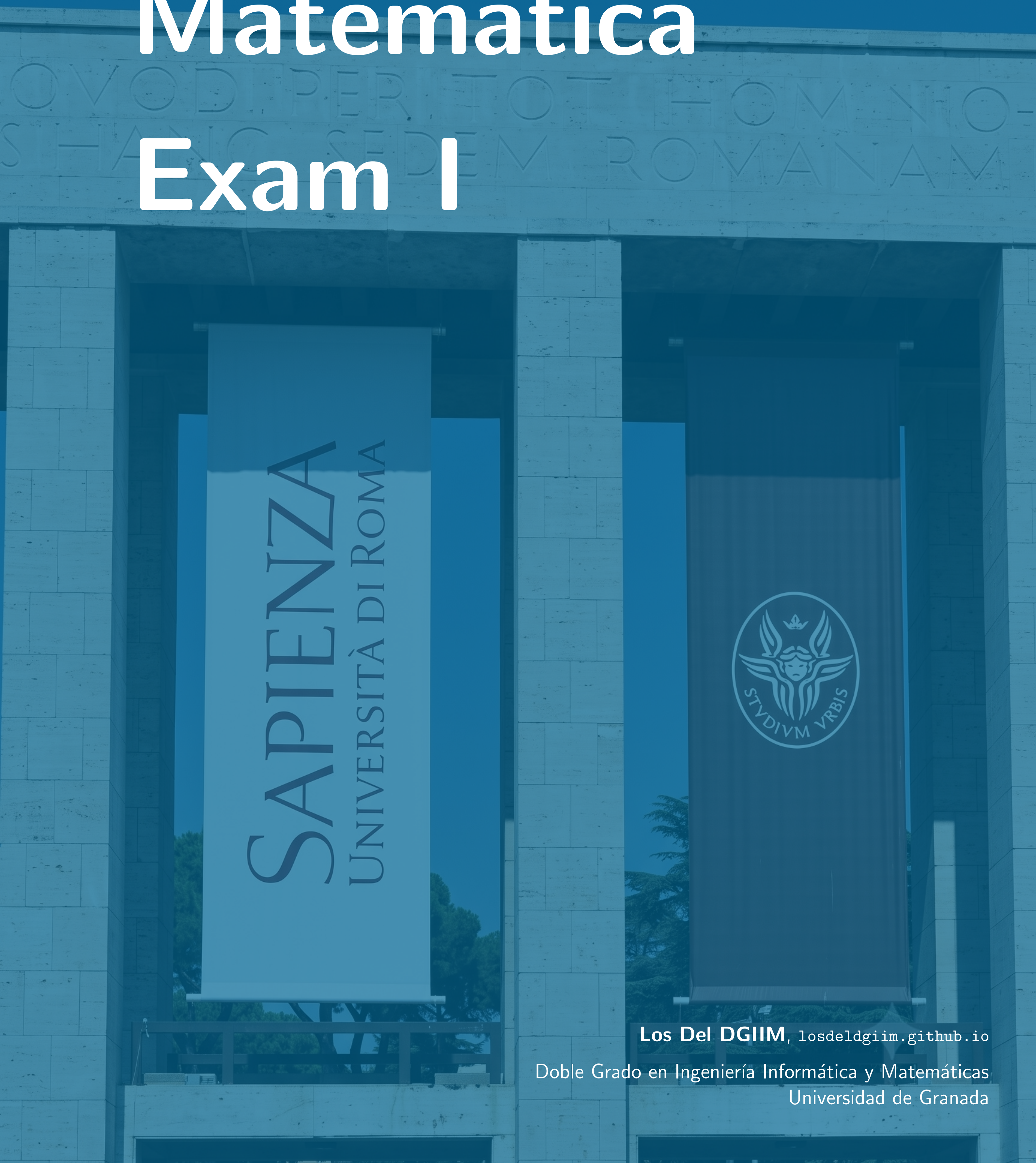
Statistica Matematica Exam I



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Doble Grado en Ingeniería Informática y Matemáticas
Universidad de Granada

Statistica Matematica Exam I

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Grado Grado en Matemáticas.

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Profesor Antonio Agresti.

Descripción Examen final escrito de Statistica Matematica.

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Duración 120 min.

Exercise 1 (9 points)**.** Let $X_1, \dots, X_n$ be a sample of independent and identically distributed random variables according to a Poisson(λ) distribution, with λ being an unknown parameter. Assume an exponential distribution with mean 1/2 as the prior distribution for λ.

- (4 points) Determine the posterior distribution for the parameter λ in terms of n and the empirical mean of the sample.
- (4 points) Determine the Bayesian estimator for λ. Show that it is asymptotically unbiased and consistent.
- (1 point) Evaluate the bias. Does there exist a value of λ for which the estimator is exactly unbiased?

Recall that

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}$$

and $\Gamma(n+1) = n!$.

Exercise 2 (6 points)**.** Let $X_1, \dots, X_n$ be a sample of independent and identically distributed random variables with density:

$$f_\theta(x) = \theta\, x^{\theta-1}, \qquad 0 < x < 1,\ \theta > 0.$$

- (3 points) Determine the maximum likelihood estimator $\hat{\theta}_{MLE}$.
- (3 points) Calculate the Fisher information for the sample.

Exercise 3 (8 points)**.** Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables. Consider: $Y_n = \left(\prod_{i=1}^n X_i\right)^{1/n}$

- Determine $\lim_{n\to\infty} Y_n$ almost surely in the following cases:
  - (3 points) X_n is uniformly distributed on $(0,1)$.
  - (2 points) X_n is distributed as e^Z where $P(Z = k) = \dfrac{1}{2^k}$ for $k \in \mathbb{N}$ (Geometric).
- (3 points) In the case where X_n is uniformly distributed on $(0,1)$, determine the limit in distribution of:

$$\sqrt{n}(Y_n - e^{-1}).$$

Hint for the last point: Use the fact that if a sequence of random variables $(W_n)_n$ satisfies

$$\sqrt{n}(W_n - \theta) \xrightarrow{d} N(0, \sigma^2)$$

in distribution, then

$$\sqrt{n}(f(W_n) - f(\theta)) \xrightarrow{d} N(0, \sigma^2[f'(\theta)]^2)$$

in distribution, for any continuously differentiable function f.

Exercise 4 (7 points)**.** Let $(\mathbb{R}^2_+, \mathcal{B}(\mathbb{R}^2_+), \mathrm{Unif}_{[0,a]} \otimes \mathrm{Unif}_{[0,b]}, (a,b) \in \mathbb{R}^2_+)$ be a statistical model.

- Identify a Pivot distribution for the parameter $\theta = (a,b)$ and determine a (non-trivial) confidence region of level $\gamma \in (0,1)$.
- Let $\tau(a,b) = (a+b, a-b)$ be a characteristic of the model. Determine a Pivot distribution for τ and determine a (non-trivial) confidence region of level $\gamma \in (0,1)$.